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PAPER-1(B.E./B. TECH.)

JEE (Main) 2021

Questions & Solutions

Date : 26 February, 2021 (SHIFT-2) Time ; (3.00 pm to 06.00 pm)

Duration : 3 Hours | Max. Marks : 300

SUBJECT : MATHEMATICS

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MATHEMATICS

SECTION A

1. If vectors $\vec{a}_1 = x\hat{i} - \hat{j} + \hat{k}$ and $\vec{a}_2 = \hat{i} + y\hat{j} + z\hat{k}$ are collinear, then a possible unit vector parallel to the vector $x\hat{i} + y\hat{j} + z\hat{k}$ is

(1) $\frac{1}{\sqrt{2}}\left(-\hat{j} + \hat{k}\right)$ (2) $\frac{1}{\sqrt{2}}\left(\hat{i} - \hat{j}\right)$ (3) $\frac{1}{\sqrt{3}}\left(\hat{i} + \hat{j} - \hat{k}\right)$ (4) $\frac{1}{\sqrt{3}}\left(\hat{i} - \hat{j} + \hat{k}\right)$

Ans. (4)

Sol. $\frac{x}{1} = -\frac{1}{y} = \frac{1}{z} = \lambda(\text{let})$

Unit vector parallel to $x\hat{i} + y\hat{j} + z\hat{k} = \pm \frac{\left(\lambda\hat{i} - \frac{1}{\lambda}\hat{j} + \frac{1}{\lambda}\hat{k}\right)}{\sqrt{\lambda^2 + \frac{2}{\lambda^2}}}$

for $\lambda = 1$ it is $\pm \frac{(\hat{i} - \hat{j} + \hat{k})}{\sqrt{3}}$

2. Let A = {1, 2, 3, ..., 10} and $f : A \to A$ be defined as $f(k) = \begin{cases} k+1 & \text{if k is odd} \\ k & \text{if k is even} \end{cases}$ Then the number of possible functions $g : A \to A$ such that $gof = f$ is

(1) 10^5 (2) $^{10}C_5$ (3) 5^5 (4) 5!

Ans. (1)

Sol. g(f(x)) = f(x)

$\Rightarrow$ g(x) = x, when x is even.

$\therefore$ So total number of functions from A to A

$= 10^5 \times 1 = 10^5$

3. Let $f : R \to R$ be defined as $f(x) = \begin{cases} 2\sin\left(-\frac{\pi x}{2}\right), & \text{if } x < -1 \\ |ax^2 + x + b|, & \text{if } -1 \le x \le 1 \\ \sin(\pi x), & \text{if } x > 1 \end{cases}$ If $f(x)$ is continuous on R, then a + b equals:

(1) –3 (2) –1 (3) 3 (4) 1

Ans. (2)

Sol. If f is continuous at x = –1, then

$f(-1^-) = f(-1)$

$\Rightarrow 2 = |a - 1 + b|$

$\Rightarrow |a + b - 1| = 2$ ..... (i)

similarly

$f(1^-) = f(1)$

$\Rightarrow |a + b + 1| = 0$

$\Rightarrow a + b = -1$

4. For x > 0, if $f(x)=\int_1^x \frac{\log_e t}{(1+t)}dt$, then $f(e)+f\left(\frac{1}{e}\right)$ is equal to

(1) 1 (2) –1 (3) $\frac{1}{2}$ (4) 0

Ans. (3)

Sol. $f(e) + f\left(\frac{1}{e}\right) = \int_1^e \frac{\ell nt}{1+t}\,dt + \int_1^{1/e} \frac{\ell nt}{1+t}\,dt = I_1 + I_2$

$I_2 = \int_1^{1/e} \frac{\ell nt}{1+t}\,dt$ put $t = \frac{1}{z}$ $dt = -\frac{dz}{z^2}$

$= \int_1^e -\frac{\ell nz}{1+\frac{1}{z}} \times \left(-\frac{dz}{z^2}\right) = \int_1^e \frac{\ell nz}{z(z+1)}dz$

$f(e) + f\left(\frac{1}{e}\right) = \int_1^e \frac{\ell nt}{1+t}\,dt + \int_1^e \frac{\ell nt}{t(t+1)}\,dt = \int_1^e \frac{\ell nt}{1+t} + \frac{\ell nt}{t(t+1)}\,dt$

$= \int_1^e \frac{\ell nt}{t}dt = \int_0^1 u\,du = \left.\frac{u^2}{2}\right|_0^1 = \frac{1}{2}$

5. A natural number has prime factorization given by n = $2^x3^y5^z$, where y and z are such that y + z = 5 and $y^{-1} + z^{-1} = \frac{5}{6}$, y > z. Then the number of odd divisors of n, including 1, is :

(1) 11 (2) 6 (3) 6x (4) 12

Ans. (4)

Sol. Solving given two equation we get y = 3, z = 2

$\Rightarrow N = 2^x\, 3^3\, 5^2$

number of odd divisor = (2 + 1) (3 + 1) = 12

6. Let $f(x) = \sin^{-1}x$ and $g(x) = \frac{x^2-x-2}{2x^2-x-6}$. If $g(2) = \lim_{x\to 2} g(x)$, then the domain of the function $f$og is :

(1) $(-\infty, -2]\cup\left[-\frac{3}{2}, \infty\right)$ (2) $(-\infty, -2]\cup[-1, \infty)$

(3) $(-\infty, -2]\cup\left[-\frac{4}{3}, \infty\right)$ (4) $(-\infty, -1]\cup[2, \infty)$

Ans. (3)

Sol. $g(2) = \lim_{x\to 2} \frac{(x-2)(x+1)}{(2x+3)(x-2)} = \frac{3}{7}$

For domain of fog (x)

$\left|\frac{x^2-x-2}{2x^2-x-6}\right| \le 1 \quad \Rightarrow \quad (x+1)^2 \le (2x+3)^2$

$\Rightarrow (3x+4)(x+2) \ge 0$

$x \in (-\infty, -2] \cup \left(-\frac{4}{3}, \infty\right]$

7. The triangle of maximum area that can be inscribed in a given circle of radius 'r' is :

(1) An isosceles triangle with base equal to 2r.

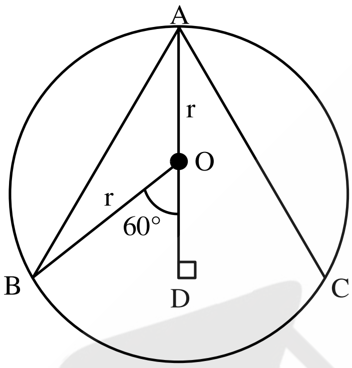
(2) An equilateral triangle of height $\frac{2r}{3}$.

(3) An equilateral triangle having each of its side of length $\sqrt{3}r$.

(4) A right angle triangle having two of its sides of length 2r and r.

Ans. (3)

Sol.



$OD = r\cos60° = \frac{r}{2}$

$\text{Height} = AD = \frac{3r}{2}$

$\text{Now } \sin 60° = \frac{3\frac{r}{2}}{AB}$

$\Rightarrow AB = \sqrt{3}\,r$

8. Let L be a line obtained from the intersection of two planes x + 2y + z = 6 and y + 2z = 4. If point P(α, β, γ) is the foot of perpendicular from (3, 2, 1) on L, then the value of 21(α + β + γ) equals :

(1) 142 (2) 68 (3) 136 (4) 102

Ans. (4)

Sol. Let D.R's of line are a, b, c

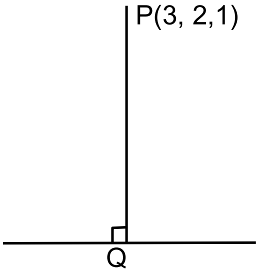
$\therefore \quad a + 2b + c = 0$

$0.a + b + 2c = 0$

$\frac{a}{3} = \frac{b}{-2} = \frac{c}{1}$

Points on the line is (–2, 4, 0)

$\therefore$ equation of line is $\frac{x+2}{3} = \frac{y-4}{-2} = \frac{z}{1} = 1$



Point Q on the line is $(3\lambda - 2, -2\lambda + 4, \lambda)$

DR's of PQ ; $3\lambda - 5, -2\lambda + 2, \lambda - 1$

DR's of lines are 3, –2, 1

Since PQ $\perp$ line $\Rightarrow 3(3\lambda - 5) -2(-2\lambda+2) + 1(\lambda - 1) = 0$

$\lambda = \frac{10}{7}$

$\therefore$ Q $\left(\frac{16}{7}, \frac{8}{7}, \frac{10}{7}\right)$

9. Let $F_1(A,B,C) = (A\wedge{\sim}B) \vee [{\sim}C \wedge (A \vee B)] \vee {\sim}A$ and $F_2(A, B) = (A \vee B) \vee (B \rightarrow {\sim}A)$ be two logical expressions. Then :

(1) F_1 and F_2 both are tautologies

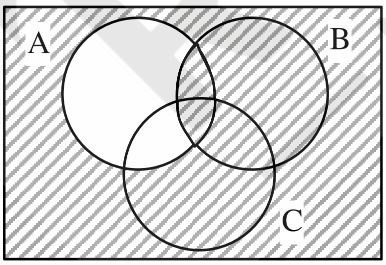
(2) F_1 is a tautology but F_2 is not a tautology

(3) F_1 is not tautology but F_2 is a tautology

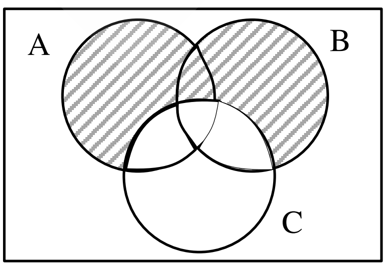
(4) Both F_1 and F_2 are not tautologies

Ans. (3)

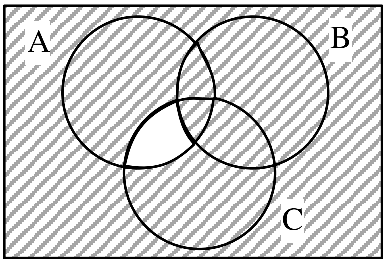
Sol. $({\sim}A \vee B) \equiv$



${\sim}C \wedge (A \vee B)$

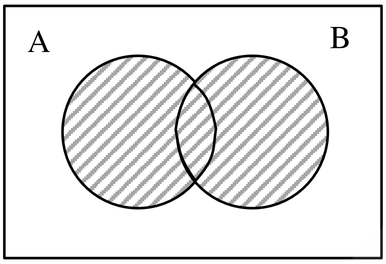


$\therefore F_1$:

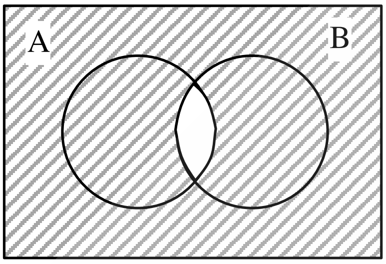


$\Rightarrow$ Not tautology

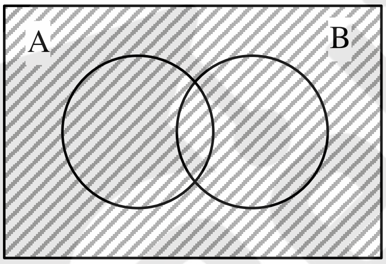
$A \vee B \equiv$



$A \to \sim B$



$\therefore F_2 \to$



Tautology

Truth table for F_1 (A, B, C)

A	B	C	~A	~C	A∨B	~A∨B	~C∧(A∨B)	(~A∨B)∨(~C∧(A∨B))∨~A
T	T	T	F	F	T	T	F	T
T	F	F	F	T	T	F	T	T
T	T	F	F	T	T	T	T	T
T	F	T	F	F	T	F	F	F
F	T	T	T	F	T	T	F	T
F	F	F	T	T	F	T	F	T
F	T	F	T	T	T	T	T	T
F	F	T	T	F	F	T	F	T

Truth table for F_2

A	B	$A\vee B$	$\sim B$	$A\to\sim B$	$(A\vee B)\vee(A\to\sim B)$
T	T	T	F	F	T
T	F	T	T	T	T
F	T	T	F	T	T
F	F	F	T	T	T

F_1 not shows tautology and F_2 shows tautology.

10. Let slope of the tangent line to a curve at any point P(x, y) be given by $\frac{xy^2+y}{x}$. If the curve intersects the line x + 2y = 4 at x = –2, then the value of y, for which the point (3, y) lies on the curve, is :

(1) $\frac{18}{35}$ (2) $-\frac{4}{3}$ (3) $-\frac{18}{19}$ (4) $-\frac{18}{11}$

Ans. (3)

Sol. $\frac{dy}{dx}=\frac{xy^2+y}{x}$

$\Rightarrow \frac{xdy-ydx}{y^2} = x\,dx$

$\Rightarrow -d\left(\frac{x}{y}\right) = d\left(\frac{x^2}{2}\right)$

$\Rightarrow \frac{-x}{y}=\frac{x^2}{2}+C$

It passes through (–2, 3)

$\Rightarrow \frac{2}{3} = 2 + C$

$\Rightarrow C = \frac{-4}{3}$

$\therefore$ curve is $\frac{-x}{y}=\frac{x^2}{2}-\frac{4}{3}$

It also passes through (3, y)

$\frac{-3}{y}=\frac{9}{2}-\frac{4}{3}$

$\Rightarrow \frac{-3}{y}=\frac{19}{6}$

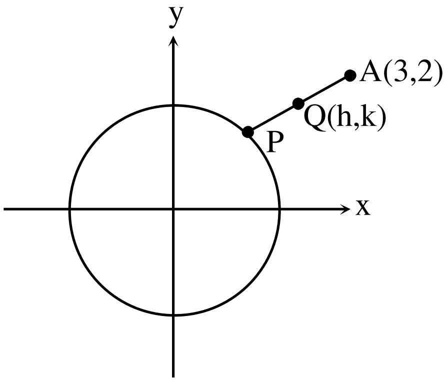
$\Rightarrow y = -\frac{18}{19}$

11. If the locus of the mid-point of the line segment from the point (3, 2) to a point on the circle, $x^2 + y^2 = 1$ is a circle of radius r, then r is equal to :

(1) 1 (2) $\frac{1}{2}$ (3) $\frac{1}{3}$ (4) $\frac{1}{4}$

Ans. (2)

Sol.



$\therefore$ P $\equiv$ (2h – 3, 2k – 2) $\rightarrow$ on circle

$\therefore \left(h-\frac{3}{2}\right)^2 + (k-1)^2 = \frac{1}{4}$

$\Rightarrow$ radius = $\frac{1}{2}$

12. Consider the following system of equations :

x + 2y – 3z = a

2x + 6y – 11z = b

x – 2y + 7z = c,

where a, b and c are real constants. Then the system of equations :

(1) has a unique solution when 5a = 2b + c

(2) has infinite number of solutions when 5a = 2b + c

(3) has no solution for all a, b and c

(4) has a unique solution for all a, b and c

Ans. (2)

Sol. $D = \begin{vmatrix} 1 & 2 & -3 \\ 2 & 6 & -11 \\ 1 & -2 & 7 \end{vmatrix}$

= 20 – 2(25) –3(–10)

= 20 – 50 + 30 = 0

$D_1 = \begin{vmatrix} a & 2 & -3 \\ b & 6 & -11 \\ c & -2 & 7 \end{vmatrix}$

= 20a – 2(7b + 11c) –3(–2b – 6c)

$= 20a - 14b - 22c + 6b + 18c$

$= 20a - 8b - 4c$

$= 4(5a - 2b - c)$

$$D_2 = \begin{vmatrix} 1 & a & -3 \\ 2 & b & -11 \\ 1 & c & 7 \end{vmatrix}$$

$= 7b + 11c - a(25) - 3(2c - b)$

$= 7b + 11c - 25a - 6c + 3b$

$= -25a + 10b + 5c$

$= -5(5a - 2b - c)$

$$D_3 = \begin{vmatrix} 1 & 2 & a \\ 2 & 6 & b \\ 1 & -2 & c \end{vmatrix}$$

$= 6c + 2b - 2(2c - b) - 10a$

$= -10a + 4b + 2c$

$= -2(5a - 2b - c)$

for infinite solution

$D = D_1 = D_2 = D_3 = 0$

$\Rightarrow 5a = 2b + c$

13. If $0 < a, b < 1$, and $\tan^{-1}a + \tan^{-1}b = \frac{\pi}{4}$, then the value of

$(a+b) - \left(\frac{a^2+b^2}{2}\right) + \left(\frac{a^3+b^3}{3}\right) - \left(\frac{a^4+b^4}{4}\right) + \ldots$ is :

(1) $\log_e 2$ (2) $e^2 - 1$ (3) e (4) $\log_e\left(\frac{e}{2}\right)$

Ans. (1)

Sol. $\tan^{-1}\left(\frac{a+b}{1-ab}\right) = \frac{\pi}{4} \Rightarrow a + b = 1 - ab \Rightarrow (1 + a)(1 + b) = 2$

Now, $a + b - \left(\frac{a^2+b^2}{2}\right) + \left(\frac{a^3+b^3}{3}\right) \ldots\ldots \infty$

$= \left(a - \frac{a^2}{2} + \frac{a^3}{3} \ldots\ldots\right) + \left(b - \frac{b^2}{2} + \frac{b^3}{3} \ldots\ldots\right)$

$= \ell n(1 + a) + \ell n(1 + b) = \ell n(1 + a)(1 + b) = \ell n\, 2$

14. The sum of the series $\sum_{n=1}^{\infty} \frac{n^2+6n+10}{(2n+1)!}$ is equal to :

(1) $\frac{41}{8}e + \frac{19}{8}e^{-1} - 10$ (2) $\frac{41}{8}e - \frac{19}{8}e^{-1} - 10$ (3) $\frac{41}{8}e + \frac{19}{8}e^{-1} + 10$ (4) $-\frac{41}{8}e + \frac{19}{8}e^{-1} - 10$

Ans. (2)

Sol. $\sum_{n=1}^{\infty} \frac{n^2+6n+10}{(2n+1)!}$

put 2n + 1 = r, where r = 3,5,7,.........

$\Rightarrow n = \frac{r-1}{2}$

$$\frac{n^2-6n+10}{(2n+1)!} = \frac{\left(\frac{r-1}{2}\right)^2 + 3r - 3 + 10}{r!} = \frac{r^2+10r+29}{4r!}$$

Now $\sum_{r=3,5,7} \frac{r(r-1)+11r+29}{4r!} = \frac{1}{4} \sum_{r=3,5,7,........} \left(\frac{1}{(r-2)!} + \frac{11}{(r-1)!} + \frac{29}{r!} \right)$

$$= \frac{1}{4}\left\{\left(\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+........\right)+11\left(\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+........\right)+29\left(\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+........\right)\right\}$$

$$= \frac{1}{4}\left\{\frac{e-\frac{1}{e}}{2}+11\left(\frac{e+\frac{1}{e}-2}{2}\right)+29\left(\frac{e-\frac{1}{e}-2}{2}\right)\right\}$$

$$= \frac{1}{8}\left\{e-\frac{1}{e}+11e+\frac{11}{e}-22+29e-\frac{29}{e}-58\right\}$$

$$= \frac{1}{8}\left\{41e-\frac{19}{e}-80\right\}$$

15. Let $f(x)$ be a differentiable function at x = a with $f'(a) = 2$ and $f(a) = 4$. Then $\lim_{x \to a} \frac{xf(a) - af(x)}{x-a}$ equals :

(1) 2a + 4 (2) 4 – 2a (3) 2a – 4 (4) a + 4

Ans. (2)

Sol. By L–H rule

$$L = \lim_{x \to a} \frac{f(a) - af'(x)}{1}$$

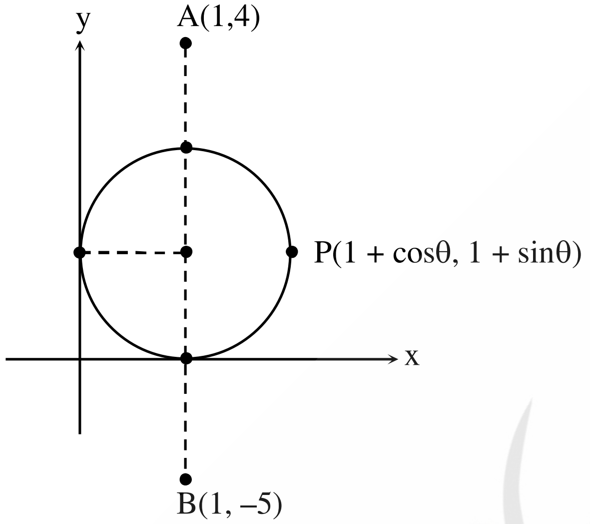
$\therefore$ L = 4 – a (2)

16. Let A(1, 4) and B(1, –5) be two points. Let P be a point on the circle $(x-1)^2 + (y-1)^2 = 1$ such that $(PA)^2 + (PB)^2$ have maximum value, then the points P, A and B lie on :

(1) a straight line (2) a hyperbola (3) an ellipse (4) a parabola

Ans. (1)

Sol.



$\therefore\ PA^2 = \cos^2\theta + (\sin\theta - 3)^2 = 10 - 6\sin\theta$

$PB^2 = \cos^2\theta + (\sin\theta + 6)^2 = 37 + 12\sin\theta$

$PA^2 + PB^2 = 47 + 6\sin\theta|_{max.} \Rightarrow \theta = \frac{\pi}{2}$

$\therefore$ P, A, B lie on a line x = 1

17. If the mirror image of the point (1, 3, 5) with respect to the plane $4x - 5y + 2z = 8$ is (α, β, γ), then $5(\alpha + \beta + \gamma)$ equals :

(1) 47 (2) 43 (3) 39 (4) 41

Ans. (1)

Sol. $\frac{x-1}{4} = \frac{y-3}{-5} = \frac{z-5}{2} = \frac{18}{45} = \frac{2}{5}$

$x - 1 = \frac{8}{5} \Rightarrow x = \frac{13}{5}$

$y = 1$

$z = \frac{29}{5}$

$\alpha = \frac{13}{5}, \beta = 1, \gamma = \frac{29}{5}$

$5(\alpha + \beta + \gamma) = 47$

18. Let $f(x)=\int_0^x e^t f(t)dt+e^x$ be a differentiable function for all x ∈ R. Then $f(x)$ equals :

(1) $2e^{(e^x-1)}-1$ (2) $e^{e^x}-1$ (3) $2e^{e^x}-1$ (4) $e^{(e^x-1)}$

Ans. (1)

Sol. $f'(x) = e^x . f(x) + e^x$

$\Rightarrow \frac{f'(x)}{f(x)+1} = e^x \quad \Rightarrow \ell n\,(f(x)+1) = e^x + c$

put x = 0

$\ell n\,2 = 1 + c$

$\therefore \ \ell n\,(f(x)+1) = e^x + \ell n\,2 - 1$

$\Rightarrow f(x) + 1 = 2 . e^{e^x-1} \quad \Rightarrow f(x) = 2e^{e^x-1} - 1$

19. Let A_1 be the area of the region bounded by the curves y = sinx, y = cosx and y-axis in the first quadrant. Also, let A_2 be the area of the region bounded by the curves y = sinx, y = cosx, x-axis and $x=\frac{\pi}{2}$ in the first quadrant. Then,

(1) $A_1 : A_2 = 1:\sqrt{2}$ and $A_1 + A_2 = 1$

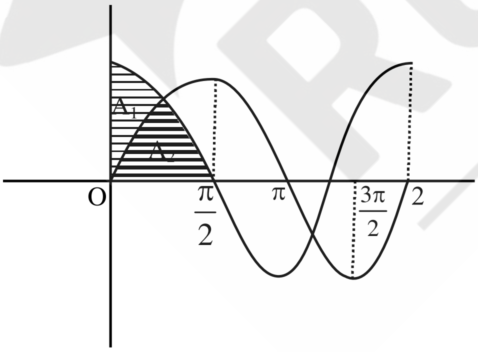
(2) $A_1 = A_2$ and $A_1 + A_2 = \sqrt{2}$

(3) $2A_1 = A_2$ and $A_1 + A_2 = 1+\sqrt{2}$

(4) $A_1 : A_2 = 1:2$ and $A_1 + A_2 = 1$

Ans. (1)

Sol. $A_1 + A_2 = \int_0^{\pi/2} \cos x\,.\,dx = \sin x\Big|_0^{\pi/2} = 1$



$A_1 = \int_0^{\pi/4} (\cos x - \sin x)\,dx = (\sin x + \cos x)\Big|_0^{\pi/4} = \sqrt{2} - 1$

$\therefore \ A_2 = 1 - (\sqrt{2}-1) = 2 - \sqrt{2}$

$\therefore \ \frac{A_1}{A_2} = \frac{\sqrt{2}-1}{\sqrt{2}(\sqrt{2}-1)} = 1 : \sqrt{2}$

20. A seven digit number is formed using digits 3, 3, 4, 4, 4, 5, 5. The probability, that number so formed is divisible by 2, is :

(1) $\frac{6}{7}$ (2) $\frac{1}{7}$ (3) $\frac{3}{7}$ (4) $\frac{4}{7}$

Ans. (3)

Sol. $n(S) = \frac{7!}{2!3!2!}$

$n(E) = \frac{6!}{2!2!2!}$

$P(E) = \frac{n(E)}{n(S)} = \frac{6!}{7!} \times \frac{2!3!2!}{2!2!2!}$

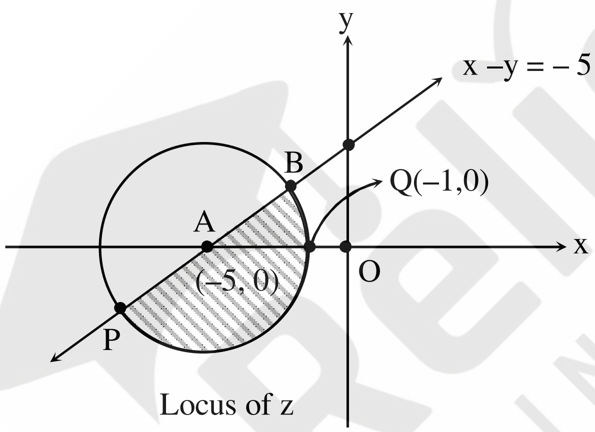
$= \frac{1}{7} \times 3 = \frac{3}{7}$

SECTION B

1. Let z be those complex numbers which satisfy $|z + 5| \le 4$ and $z(1+i)+\bar{z}(1-i) \ge -10, i = \sqrt{-1}$. If the maximum value of $|z + 1|^2$ is $\alpha + \beta\sqrt{2}$, then the value of $(\alpha + \beta)$ is _______.

Ans. 48

Sol.



$\therefore$ $P \equiv (-5 - 2\sqrt{2}, -2\sqrt{2})$

$\therefore$ $(PQ)^2\Big|_{max} = 32 + 16\sqrt{2}$

$\alpha = 32$

$\beta = 16$

$\alpha + \beta = 48$

2. Let the normal at all the points on a given curve pass through a fixed point (a, b). If the curve passes through (3, –3) and $(4, -2\sqrt{2})$, and given that $a - 2\sqrt{2}\, b = 3$, then $(a^2 + b^2 + ab)$ is equal to ______.

Ans. 9

Sol. Let the equation of normal is $Y - y = -\frac{1}{m}(X - x)$

Satisfy (a, b) in it $b - y = -\frac{1}{m}(a - x)$

$\Rightarrow (b - y)\,dy = (x - a)\,dx$

by $-\frac{y^2}{2} = \frac{x^2}{2} - ax + c$ ……(i)

It passes through (3, –3) & $(4, -2\sqrt{2})$

$\therefore\ -3b - \frac{9}{2} = \frac{9}{2} - 3a + c$

$\Rightarrow -6b - 9 = 9 - 6a + 2c$

$\Rightarrow 6a - 6b - 2c = 18$

$\Rightarrow 3a - 3b - c = 9$ ……(ii)

Also

$-2\sqrt{2}\,b - 4 = 8 - 4a + c$

$4a - 2\sqrt{2}\,b - c = 12$ ……(iii)

Also $a - 2\sqrt{2}\,b = 3$ ……(iv) (given)

(ii) – (iii) $\Rightarrow -a + (2\sqrt{2} - 3)\,b = -3$ ……(v)

(iv) + (v) $\Rightarrow$ b = 0 a = 3

$\therefore\ a^2 + b^2 + ab = 9$

3. Let α and β be two real numbers such that $\alpha + \beta = 1$ and $\alpha\beta = -1$. Let $p_n = (\alpha)^n + (\beta)^n$, $p_{n-1} = 11$ and $p_{n+1} = 29$ for some integer $n \ge 1$. Then, the value of p_n^2 is ______.

Ans. 324

Sol. Quadratic Equation whose roots are α, β : $x^2 - x - 1 = 0$

$\therefore\ \alpha^2 = \alpha + 1 \Rightarrow \alpha^n = \alpha^{n-1} + \alpha^{n-2}$

$\beta^2 = \beta + 1 \Rightarrow \beta^n = \beta^{n-1} + \beta^{n-2}$

$\therefore\ P_n = P_{n-1} + P_{n-2}$

$\Rightarrow P_{n+1} = P_n + P_{n-1}$

$\Rightarrow 29 = P_n + 11 \Rightarrow P_n = 18$

$\Rightarrow (Pn)^2 = 324$

4. If $I_{m,n} = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$, for $m, n \ge 1$ and $\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\,dx = \alpha\, I_{m,n}$, $\alpha \in R$, then α equals ____.

Ans. 1

Sol. $I_{m,n} = \int_0^1 x^{m-1}.(1-x)^{n-1}\,dx$ put $x = \dfrac{1}{y+1}$

$$I_{m,n} = \int_\infty^0 \frac{y^{n-1}}{(y+1)^{m+n}}(-1)\,dy = \int_0^\infty \frac{y^{n-1}}{(y+1)^{m+n}}\,dy \quad .....(i)$$

Similarly $I_{m,n} = \int_0^1 x^{n-1}.(1-x)^{m-1}\,dx$

$$\Rightarrow I_{m,n} = \int_0^\infty \frac{y^{m-1}}{(y+1)^{m+n}}\,dy \quad .....(ii)$$

From (i) & (ii)

$$2I_{m,n} = \int_0^\infty \frac{y^{m-1}+y^{n-1}}{(y+1)^{m+n}}\,dy$$

$$\Rightarrow 2I_{m,n} = \int_0^1 \frac{y^{m-1}+y^{n-1}}{(y+1)^{m+n}}\,dy + \int_1^\infty \frac{y^{m-1}+y^{n-1}}{(y+1)^{m+n}}\,dy$$

Put $y = \dfrac{1}{z}$

$$\Rightarrow 2I_{m,n} = \int_0^1 \frac{y^{m-1}+y^{n-1}}{(y+1)^{m+n}}\,dy + \int_1^\infty \frac{z^{m-1}+z^{n-1}}{(z+1)^{m+n}}\,dz$$

$$\Rightarrow I_{m,n} = \int_0^1 \frac{y^{m-1}+y^{n-1}}{(y+1)^{m+1}}\,dy \Rightarrow \alpha = 1$$

5. If the arithmetic mean and geometric mean of the p^{th} and q^{th} terms of the sequence–16, 8, –4, 2, ... satisfy the equation $4x^2 - 9x + 5 = 0$, then p + q is equal to ______.

Ans. 10

Sol. –16, 8, –4, 2 ......

p^{th} term $t_p = -16\left(\dfrac{-1}{2}\right)^{p-1}$

q^{th} term $t_q = -16\left(\dfrac{-1}{2}\right)^{q-1}$

Now $\dfrac{t_p+t_q}{2} = \dfrac{5}{4}$ & $\sqrt{t_p t_q} = 1$

$\Rightarrow 16^2\left(-\dfrac{1}{2}\right)^{p+q-2} = 1$

$\Rightarrow 2^8 = (-2)^{(p+q-2)}$

$\Rightarrow p + q = 10$

6. The total number of 4-digit numbers whose greatest common divisor with 18 is 3, is ______.

Ans. 1000

Sol. Number must be an odd multiple of 3 and not a multiple of 9

4-digit odd multiples of 3 are

1005, 1011, ……., 9999 $\rightarrow$ 1499

4-digit odd multiples of 9 are

1017, 1035, ……., 9999 $\rightarrow$ 499

$\therefore$ Required numbers $\rightarrow$ 1000

7. Let L be a common tangent line to the curves $4x^2 + 9y^2 = 36$ and $(2x)^2 + (2y)^2 = 31$. Then the square of the slope of the line L is ______.

Ans. 3

Sol. $E : \frac{x^2}{9} + \frac{y^2}{4} = 1$ $\qquad$ $C : x^2 + y^2 = \frac{31}{4}$

equation of tangent to ellipse

$y = mx \pm \sqrt{9m^2 + 4}$ ….(i)

equation of tangent to circle

$y = mx \pm \sqrt{\frac{31}{4}m^2 + \frac{31}{4}}$ …(ii)

Comparing equation (i) & (ii)

$9m^2 + 4 = \frac{31m^2}{4} + \frac{31}{4}$

$\Rightarrow 36m^2 + 16 = 31m^2 + 31$

$\Rightarrow 5m^2 = 15$

$\Rightarrow m^2 = 3$

8. Let a be an integer such that all the real roots of the polynomial $2x^5+5x^4 + 10x^3 + 10x^2 + 10x + 10$ lie in the interval (a, a + 1). Then, $|a|$ is equal to ______.

Ans. (2)

Sol. $f(-1) = 3 > 0$

$f(-2) = -64 + 80 - 80 + 40 - 20 + 10$

$= -34 < 0$

$\therefore$ At least one root in $(-2,-1)$

$f'(x) = 10 (x^4 + 2x^3 + 3x^2 + 2x + 1)$

$= 10\left(x^2 + \frac{1}{x^2} + 2\left(x + \frac{1}{x}\right) + 3\right)$

$= 10\left(\left(x + \frac{1}{x}\right)^2 + 2\left(x + \frac{1}{x}\right) + 1\right)$

$= 10\left(\left(x + \frac{1}{x}\right) + 1\right)^2 > 0; \forall x \in R$

$\therefore$ Exactly one real root in (–2,–1)

9. Let X_1, X_2, ..., X_{18} be eighteen observations such that $\sum_{i=1}^{18}(X_i - \alpha) = 36$ and $\sum_{i=1}^{18}(X_i - \beta)^2 = 90$, where α and β are distinct real numbers. If the standard deviation of these observations is 1, then the value of $|\alpha - \beta|$ is ______.

Ans. 4

Sol. $\Sigma x_i - 18\alpha = 36$

$\Sigma x_i = 18(\alpha + 2)$ …(i)

$\Sigma x_i^2 + 18\beta^2 - 2\beta\, \Sigma x_i = 90$

$\Sigma x_i^2 + 18\beta^2 - 2\beta \times 18\,(\alpha + 2) = 90$

$\Sigma x_i^2 = 90 - 18\beta^2 + 36\beta\,(\alpha + 2)$ …(ii)

$\sigma^2 = 1 \Rightarrow \frac{1}{18}\Sigma x_i^2 - \left(\frac{\Sigma x_i}{18}\right)^2 = 1$

$\Rightarrow \frac{1}{18}\,(90 - 18\beta^2 + 36\alpha\beta + 72\beta) - \left(\frac{18(\alpha + 2)}{18}\right)^2 = 1$

$\Rightarrow 90 - 18\beta^2 + 36\alpha\beta + 72\beta - 18(\alpha + 2)^2 = 18$

$\Rightarrow 5 - \beta^2 + 2\alpha\beta + 4\beta - (\alpha + 2)^2 = 1$

$\Rightarrow 5 - \beta^2 + 2\alpha\beta + 4\beta - \alpha^2 - 4 - 4\alpha = 1$

$-\alpha^2 - \beta^2 + 2\alpha\beta + 4\beta - 4\alpha = 0$

$- \quad (\alpha - \beta)^2 - 4(\alpha - \beta) = 0$

$-(\alpha - \beta)(\alpha - \beta + 4) = 0$

$\Rightarrow \alpha - \beta = -4$ $(\alpha \neq \beta)$

$|\beta - \alpha| = 4$

10. If the matrix $A=\begin{bmatrix}1&0&0\\0&2&0\\3&0&-1\end{bmatrix}$ satisfies the equation $A^{20}+\alpha A^{19}+\beta A=\begin{bmatrix}1&0&0\\0&4&0\\0&0&1\end{bmatrix}$ for some real numbers α and β, then $\beta-\alpha$ is equal to ______.

Ans. 4

Sol. $$A^2=\begin{bmatrix}1&0&0\\0&2&0\\3&0&-1\end{bmatrix}\begin{bmatrix}1&0&0\\0&2&0\\3&0&-1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&4&0\\0&0&1\end{bmatrix}$$

$$A^3=\begin{bmatrix}1&0&0\\0&4&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&2&0\\3&0&-1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&8&0\\3&0&-1\end{bmatrix}$$

$$A^4=\begin{bmatrix}1&0&0\\0&4&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&4&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&16&0\\0&0&1\end{bmatrix}$$

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$$A^{19}=\begin{bmatrix}1&0&0\\0&2^{19}&0\\3&0&-1\end{bmatrix}, A^{20}=\begin{bmatrix}1&0&0\\0&2^{20}&0\\0&0&1\end{bmatrix}$$

$$\text{L.H.S} = A^{20}+\alpha A^{19}+\beta A=\begin{bmatrix}1+\alpha+\beta&0&0\\0&2^{20}+\alpha 2^{19}+2\beta&0\\3\alpha+3\beta&0&1-\alpha-\beta\end{bmatrix}$$

$$\text{R.H.S}=\begin{bmatrix}1&0&0\\0&4&0\\0&0&1\end{bmatrix}\Rightarrow \alpha+\beta=0 \text{ and}$$

$$2^{20}+\alpha 2^{19}+2\beta=4$$

$$2^{20}+\alpha(2^{19}-2)=4$$

$\alpha=-2$

$\beta=2\Rightarrow(\beta-\alpha)=4$