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PAPER-1(B.E./B. TECH.)

JEE (Main) 2021

Questions & Solutions

Date : 25 February, 2021 (SHIFT-1) Time ; (9.00 am to 12.00 pm)

Duration : 3 Hours | Max. Marks : 300

SUBJECT : MATHEMATICS

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MATHEMATICS

SECTION-A

1. When a missile is fired from a ship, the probability that it is intercepted is $\frac{1}{3}$ and the probability that the missile hits the target, given that it is not intercepted, is $\frac{3}{4}$. If three missiles are fired independently from the ship, then the probability that all three hit the target, is :

(1) $\frac{1}{27}$ (2) $\frac{3}{4}$ (3) $\frac{1}{8}$ (4) $\frac{3}{8}$

Ans. (3)

Sol. Prob. $= \left(\frac{2}{3}.\frac{3}{4}\right)^3 = \frac{1}{8}$

2. If $0 < \theta, \phi < \frac{\pi}{2}, x = \sum_{n=0}^{\infty} \cos^{2n}\theta,\ y = \sum_{n=0}^{\infty} \sin^{2n}\phi$ and $z = \sum_{n=0}^{\infty} \cos^{2n}\theta \cdot \sin^{2n}\phi$ then :

(1) $xy - z = (x + y)\, z$
(2) $xy + yz + zx = z$
(3) $xyz = 4$
(4) $xy + z = (x + y)z$

Ans. (4)

Sol. $x = 1 + \cos^2\theta + \cos^4\theta + \text{............} = \frac{1}{1-\cos^2\theta} = \frac{1}{\sin^2\theta}$

$y = 1 + \sin^2\phi + \sin^4\phi + \text{............} = \frac{1}{1-\sin^2\phi} = \frac{1}{\cos^2\phi}$

$z = 1 + \sin^2\phi\cos^2\theta + \sin^4\phi\cos^4\theta + \text{............} = \frac{1}{1-\sin^2\phi\cos^2\theta} = \frac{1}{1-\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)}$

$= \frac{1}{\frac{1}{x}+\frac{1}{y}-\frac{1}{xy}} = z$

$xy = z(x + y) - z$

$xy + z = (x + y)z$

3. Let f, g : N $\to$ N such that $f(n + 1) = f(n) + f(1)\ \forall\ n \in N$ and g be any arbitrary function. Which of the following statements is NOT true?

(1) If fog is one-one, then g is one-one
(2) If f is onto, then $f(n) = n\ \forall\ n \in N$
(3) f is one-one
(4) If g is onto, then fog is one-one

Ans. (4)

Sol. $f(n + 1) = f(n) + 1$
$f(2) = 2f(1)$
$f(3) = 3f(1)$
$f(4) = 4f(1)$
….
$f(n) = nf(1)$
f(x) is one-one

4. The equation of the line through the point (0,1,2) and perpendicular to the line $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{-2}$ is :

(1) $\frac{x}{3}=\frac{y-1}{4}=\frac{z-2}{3}$ (2) $\frac{x}{3}=\frac{y-1}{-4}=\frac{z-2}{3}$ (3) $\frac{x}{3}=\frac{y-1}{4}=\frac{z-2}{-3}$ (4) $\frac{x}{-3}=\frac{y-1}{4}=\frac{z-2}{3}$

Ans. (4)

Sol. Direction ratio of given line 2, 3, – 2
there are infinite lines which passes through (1, 1, 2) and perpendicular to given line
according to options (4) is correct

5. Let α be the angle between the lines whose direction cosines satisfy the equations $l + m - n = 0$ and $l^2 + m^2 - n^2 = 0$. Then the value of $\sin^4\alpha + \cos^4\alpha$ is :

(1) $\frac{3}{4}$ (2) $\frac{3}{8}$ (3) $\frac{5}{8}$ (4) $\frac{1}{2}$

Ans. (3)

Sol. $l^2 + m^2 + n^2 = 1$

$\therefore\ 2n^2 = 1 \Rightarrow n = \pm\frac{1}{\sqrt{2}}$

$\therefore\ l^2 + m^2 = \frac{1}{2}\ \&\ l + m = \frac{1}{\sqrt{2}} \quad \Rightarrow \frac{1}{2} - 2\,l\,m = \frac{1}{2}$

$\Rightarrow l\,m = 0$ or $m = 0$

$\therefore\ l = 0, m = \frac{1}{\sqrt{2}}$ or $l = \frac{1}{\sqrt{2}}$

$< 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} >$ or $< \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} >$

$\therefore\ \cos\alpha = 0 + 0 + \frac{1}{2} = \frac{1}{2}$

$\therefore\ \sin^4\alpha + \cos^4\alpha = 1 - \frac{1}{2}\sin^2(2\alpha) = 1 - \frac{1}{2}.\frac{3}{4} = \frac{5}{8}$

6. The value of the integral $\int \frac{\sin\theta.\sin 2\theta(\sin^6\theta+\sin^4\theta+\sin^2\theta)\sqrt{2\sin^4\theta+3\sin^2\theta+6}}{1-\cos 2\theta}d\theta$ is :

(where c is a constant of integration)

(1) $\frac{1}{18}\left[11-18\sin^2\theta+9\sin^4\theta-2\sin^6\theta\right]^{\frac{3}{2}}+c$

(2) $\frac{1}{18}\left[9-2\cos^6\theta-3\cos^4\theta-6\cos^2\theta\right]^{\frac{3}{2}}+c$

(3) $\frac{1}{18}\left[9-2\sin^6\theta-3\sin^4\theta-6\sin^2\theta\right]^{\frac{3}{2}}+c$

(4) $\frac{1}{18}\left[11-18\cos^2\theta+9\cos^4\theta-2\cos^6\theta\right]^{\frac{3}{2}}+c$

Ans. (4)

Sol. $\int \frac{2\sin^2\theta\cos 2\theta(\sin^6\theta+\sin^4\theta+\sin^2\theta)\sqrt{2\sin^4\theta+3\sin^2\theta+6}}{2\sin^2\theta}\,d\theta$

Let $\sin\theta = t$ $\cos\theta\, d\theta = dt$

$= \int (t^6+t^4+t^2)\sqrt{2t^4+3t^2+6}\ dt = \int (t^5+t^3+t)\sqrt{2t^6+3t^4+6t^2}\ dt$

Let $2t^6 + 3t^4 + 6t^2 = z$

$12(t^5 + t^3 + t)\,dt = dz$

$= \frac{1}{12}\int \sqrt{z}\ dz = \frac{1}{18}z^{3/2}+c$

$= \frac{1}{18}(2\sin^6\theta + 3\sin^4\theta + 6\sin^2\theta)^{3/2} + C$

$=\frac{1}{18}[(1-\cos^2\theta)(2(1-\cos^2\theta) + 3 - 3\cos^2\theta + 6)]^{3/2} + C$

$=\frac{1}{18}[(1\text{-}\cos^2\theta)(2\cos^4\theta - 7\cos^2\theta + 11)]^{3/2} + C$

$= \frac{1}{18}[-2\cos^6\theta + 9\cos^4\theta - 18\cos^2\theta + 11]^{3/2} + C$

7. The value of $\int_{-1}^{1} x^2 e^{[x^3]}dx$, where [t] denotes the greatest integer $\le$ t, is :

(1) $\frac{e-1}{3e}$ (2) $\frac{e+1}{3}$ (3) $\frac{e+1}{3e}$ (4) $\frac{1}{3e}$

Ans. (3)

Sol. $I = \int_{-1}^{0} x^2.e^{-1}\,dx + \int_{0}^{1} x^2\,dx$

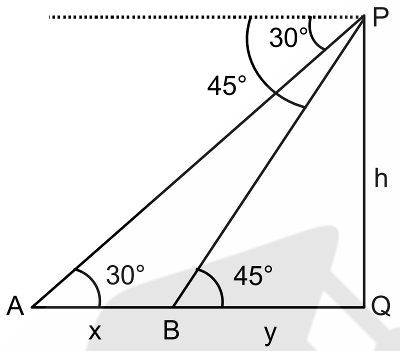
$\therefore I = \left.\frac{x^3}{3e}\right|_{-1}^{0} + \left.\frac{x^3}{3}\right|_{0}^{1} \quad \Rightarrow I = \frac{1}{3e} + \frac{1}{3}$

8. A man is observing, from the top of a tower, a boat speeding towards the tower from a certain point A, with uniform speed. At that point, angle of depression of the boat with the man's eye is 30° (Ignore man's height). After sailing for 20 seconds, towards the base of the tower (which is at the level of water), the boat has reached a point B, where the angle of depression is 45°. Then the time taken (in seconds) by the boat from B to reach the base of the tower is:

(1) 10 (2) $10\sqrt{3}$ (3) $10(\sqrt{3}+1)$ (4) $10(\sqrt{3}-1)$

Ans. (3)

Sol.



In ΔABP

$\frac{h}{x+y} = \tan 30° = \frac{1}{\sqrt{3}}$

$x + y = \sqrt{3}\,h$ …(i)

In ΔQBP

$\frac{h}{y} = \tan 45° = 1$

$h = y$ …(ii)

$x + y = \sqrt{3}\,y$

$x = (\sqrt{3} - 1)y$

Let speed is v

$\frac{x}{v} = 20 \Rightarrow x = 20\,v$

$\therefore 20\,v = (\sqrt{3} - 1)\,y$

Time to cover y distance $= \frac{y}{v} = \frac{20}{\sqrt{3}-1} = 10(\sqrt{3}+1)\,\text{sec}$

9. A tangent is drawn to the parabola $y^2 = 6x$ which is perpendicular to the line $2x + y = 1$. Which of the following points does NOT lie on it?
(1) (–6, 0) (2) (4, 5) (3) (5, 4) (4) (0, 3)

Ans. (3)

Sol. Equation of tangent : $y = mx + \frac{3}{2m}$

$m_T = \frac{1}{2}$ ($\because$ perpendicular to line $2x + y = 1$)

$\therefore$ tangent is : $y = \frac{x}{2} + 3 \Rightarrow x - 2y + 6 = 0$

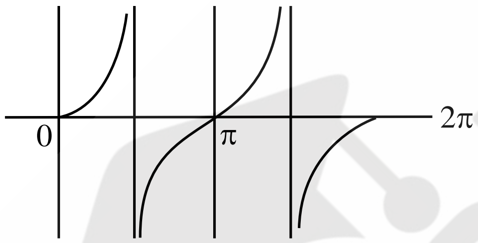
10. All possible values of $\theta \in [0, 2\pi]$ for which $\sin 2\theta + \tan 2\theta > 0$ lie in :

(1) $\left(0,\frac{\pi}{2}\right)\cup\left(\pi,\frac{3\pi}{2}\right)$ (2) $\left(0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\frac{3\pi}{4}\right)\cup\left(\pi,\frac{7\pi}{6}\right)$

(3) $\left(0,\frac{\pi}{4}\right)\cup\left(\frac{\pi}{2},\frac{3\pi}{4}\right)\cup\left(\frac{3\pi}{2},\frac{11\pi}{6}\right)$ (4) $\left(0,\frac{\pi}{4}\right)\cup\left(\frac{\pi}{2},\frac{3\pi}{4}\right)\cup\left(\pi,\frac{5\pi}{4}\right)\cup\left(\frac{3\pi}{2},\frac{7\pi}{4}\right)$

Ans. (4)

Sol.



$\tan 2\theta\,(1 + \cos 2\theta) > 0$

$2\theta \in \left(0,\frac{\pi}{2}\right) \cup \left(\pi,\frac{3\pi}{2}\right) \cup \left(2\pi,\frac{5\pi}{2}\right) \cup \left(3\pi,\frac{7\pi}{2}\right)$

$\Rightarrow \theta \in \left(0,\frac{\pi}{4}\right) \cup \left(\frac{\pi}{2},\frac{3\pi}{4}\right) \cup \left(\pi,\frac{5\pi}{4}\right) \cup \left(\frac{3\pi}{2},\frac{7\pi}{4}\right)$

11. Let the lines $(2 - i)z = (2 + i)\bar{z}$ and $(2 + i)z + (i - 2)\bar{z} - 4i = 0$, (here $i^2 = -1$) be normal to a circle C. If the line $iz + \bar{z} + 1 + i = 0$ is tangent to this circle C, then its radius is:

(1) $\frac{3}{\sqrt{2}}$ (2) $\frac{1}{2\sqrt{2}}$ (3) $3\sqrt{2}$ (4) $\frac{3}{2\sqrt{2}}$

Ans. (4)

Sol. Let $z = x + iy$ and $\bar{z} = x - iy$
$(2 - i)(x + iy) = (2 + i)(x - iy)$
$2x + 2iy - ix + y = 2x - 2iy + ix + y$
$2y = x$
$(2 + i)(x + iy) + (i - 2)(x - iy) - 4i = 0$

$2x + 2iy + ix - y + ix - 2x + y + 2iy - 4i = 0$

$4y + 2x - 4 = 0$

$2y + x - 2 = 0$

point of intersection x = 1, y = ½]

$iz + \bar{z} + 1 + i = 0$

$i(x + iy) + x - iy + 1 + i = 0$

$ix - y + x - iy + 1 + i = 0$

$i(x - y + 1) + (x - y + 1) = 0$

$x - y + 1 = 0$

$$\text{radius} = \left|\frac{1-\frac{1}{2}+1}{\sqrt{2}}\right| = \frac{3}{2\sqrt{2}}$$

12. The image of the point (3, 5) in the line $x - y + 1 = 0$, lies on :

(1) $(x-2)^2 + (y-2)^2 = 12$ (2) $(x-4)^2 + (y+2)^2 = 16$

(3) $(x-4)^2 + (y-4)^2 = 8$ (4) $(x-2)^2 + (y-4)^2 = 4$

Ans. (4)

Sol. Image of P(3,5) on the line $x - y + 1 = 0$ is

$$\frac{x-3}{1} = \frac{y-5}{-1} = \frac{-2(3-5+1)}{2} = 1$$

x = 4, y = 4

$\therefore$ Image is (4,4)

Which lies on

$(x-2)^2 + (y-4)^2 = 4$

13. If the curves, $\frac{x^2}{a} + \frac{y^2}{b} = 1$ and $\frac{x^2}{c} + \frac{y^2}{d} = 1$ intersect each other at an angle of 90°, then which of the following relations is TRUE?

(1) $a + b = c + d$ (2) $a - b = c - d$ (3) $a - c = b + d$ (4) $ab = \frac{c+d}{a+b}$

Ans. (2)

Sol. $\frac{x^2}{a} + \frac{y^2}{b} = 1$ ....(1)

diff : $\frac{2x}{a} + \frac{2y}{b}\frac{dy}{dx} = 0 \Rightarrow \frac{y}{b} + \frac{dy}{dx} = \frac{-x}{a}$

$\frac{dy}{dh} = \frac{-bx}{ay}$ ....(2)

$\frac{x^2}{c} + \frac{y^2}{d} = 1$ ....(3)

diff : $\frac{dy}{dx} = \frac{-d\ x}{cy}$ ....(4)

$m_1m_2 = -1 \Rightarrow \frac{-bx}{ay} \times \frac{-dx}{cy} = -1$

$\Rightarrow bdx^2 = -acy^2$ ....(5)

$(1) - (3) \Rightarrow \left(\frac{1}{a} - \frac{1}{c}\right)x^2 + \left(\frac{1}{b} - \frac{1}{d}\right)y^2 = 0$

$\Rightarrow \frac{c-a}{ac}x^2 + \frac{d-b}{bd} \times \left(\frac{-bd}{ac}\right)x^2 = 0$ (Using 5)

$\Rightarrow (c - a) - (d - b) = 0$

$\Rightarrow c - a = d - b$

$\Rightarrow c - d = a - b$

14. $\lim_{n\to\infty}\left(1 + \frac{1 + \frac{1}{2} + ... + \frac{1}{n}}{n^2}\right)^n$ is equal to :

(1) $\frac{1}{2}$ (2) 0 (3) $\frac{1}{e}$ (4) 1

Ans. (4)

Sol. Let limit be L

So $L = e^{\lim_{n\to\infty}\left(\frac{1+\frac{1}{2}+\frac{1}{3}+....+\frac{1}{n}}{n}\right)} = e^k$ (say)

Now assume $n = 2^p + \lambda$, $\lambda \in \{0, 1, 2, ..., 2^p - 1\}$

Now assume $1 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + ... + \left(\frac{1}{2^{p-1}} + \frac{1}{2^{p-1}+1} + ... + \frac{1}{2^p - 1}\right)$

$+ \left(\frac{1}{2^p} + \frac{1}{2^p + 1} + ... + \frac{1}{2^p + \lambda}\right) = S$

So $S < 1 + \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}\right) + ... + \underbrace{\left(\frac{1}{2^p} + \frac{1}{2^p} + ... + \frac{1}{2^p}\right)}_{(\lambda+1)\text{times}}$

$\Rightarrow S < \underbrace{1 + 1 + 1......}_{\text{p times}} + \frac{\lambda+1}{2p} < p + 1$

Hence $k \le \lim_{n\to\infty} \frac{p+1}{2^p} = 0$

Also $S > \underbrace{\left(\frac{1}{n}+\frac{1}{n}+\frac{1}{n}+.....+\frac{1}{n}\right)}_{\text{n times}} = 1$

Hence $k \ge \lim_{n\to\infty} \frac{1}{n} = 0$

So L = 1

15. The coefficients a, b and c of the quadratic equation, $ax^2 + bx + c = 0$ are obtained by throwing a dice three times. The probability that this equation has equal roots is:

(1) $\frac{1}{72}$ (2) $\frac{5}{216}$ (3) $\frac{1}{36}$ (4) $\frac{1}{54}$

Ans. (2)

Sol. $a, b, c \in \{1,2,3,4,5,6\}$

$n(s) = 6 \times 6 \times 6 = 216$

$D = 0 \Rightarrow b^2 = 4ac$

$ac = \frac{b^2}{4}$ If b = 2, ac = 1 $\Rightarrow$ a = 1, c = 1

If b = 4, ac = 4 $\Rightarrow$ a = 1, c = 4

a = 4, c = 1

a = 2, c = 2

If b = 6, ac = 9 $\Rightarrow$ a = 3, c = 3

$\therefore$ probability $= \frac{5}{216}$

16. The total number of positive integral solutions (x, y, z) such that xyz = 24 is :

(1) 36 (2) 24 (3) 45 (4) 30

Ans. (4)

Sol. $xyz = 2^3.3$

$x = 2^{r_1}.3^{k_1}$

$y = 2^{r_2}.3^{k_2}$

$z = 2^{r_3}.3^{k_3}$

$r_1 + r_2 + r_3 = 3 \Rightarrow {}^{3+3-1}C_{3-1} = {}^5C_2 = 10$

$k_1 + k_2 + k_3 = 1 \Rightarrow {}^{1+3-1}C_{3-1} = {}^3C_2 = 3$

Number of positive integral solution = $10 \times 3 = 30$

17. The integer 'k', for which the inequality $x^2 - 2(3k-1)x + 8k^2 - 7 > 0$ is valid for every x in R, is :

(1) 3 (2) 2 (3) 0 (4) 4

Ans. (1)

Sol. $D < 0$

$(2(3k-1))^2 - 4(8k^2-7) < 0$

$4(9k^2 - 6k + 1) - 4(8k^2 - 7) < 0$

$k^2 - 6k + 8 < 0$

$(k-4)(k-2) < 0$

$2 < k < 4$

then $k = 3$

18. If a curve passes through the origin and the slope of the tangent to it at any point (x, y) is $\frac{x^2-4x+y+8}{x-2}$, then this curve also passes through the point:

(1) (5, 4) (2) (4, 5) (3) (4, 4) (4) (5, 5)

Ans. (4)

Sol. $\frac{dy}{dx} = \frac{(x-2)^2+y+4}{(x-2)} = (x-2) + \frac{y+4}{(x-2)}$

Let $x - 2 = t \Rightarrow dx = dt$ and $y + 4 = u \Rightarrow dy = du$

$\frac{dy}{dx} = \frac{du}{dt}$

$\frac{du}{dt} = t + \frac{u}{t} \Rightarrow \frac{du}{dt} - \frac{u}{t} = t$

$I.F = e^{\int \frac{-1}{t}dt} = e^{-\ln t} = \frac{1}{t}$

$u.\frac{1}{t} = \int t.\frac{1}{t}dt \Rightarrow \frac{u}{t} = t + c$

$\frac{y+4}{x-2} = (x-2) + c$

Passing through (0, 0)

$c = 0$

$\Rightarrow (y+4) = (x-2)^2$

19. The statement $A \to (B \to A)$ is equivalent to :

(1) $A \to (A \wedge B)$ (2) $A \to (A \to B)$ (3) $A \to (A \leftrightarrow B)$ (4) $A \to (A \vee B)$

Ans. (4)

Sol. $\sim A \wedge (\sim B \wedge A) = t$

(1) $\sim A \vee (A \wedge B) = (\sim A \vee A) \wedge (\sim A \vee B) = \sim A \vee B$

(3) $A \to (A \leftrightarrow B) = \sim A \vee (A \leftrightarrow B) = \sim A \vee (A \to B \wedge B \to A) = \sim A \vee B$

(4) $\sim A \vee (A \vee B)$

$(\sim A \vee A) \vee (\sim A \vee B)$

$t \vee (\sim A \vee B) = t$

20. If Rolle's theorem holds for the function $f(x) = x^3 - ax^2 + bx - 4$, $x \in [1, 2]$ with $f'\left(\frac{4}{3}\right) = 0$, then ordered pair (a, b) is equal to :
(1) (5, 8) (2) (–5, 8) (3) (5, –8) (4) (–5, –8)

Ans. (1)

Sol. $f(1) = f(2)$

$\Rightarrow 1 - a + b - 4 = 8 - 4a + 2b - 4$

$3a - b = 7$ ...... (1)

$f'(x) = 3x^2 - 2ax + b$

$\Rightarrow f'\left(\frac{4}{3}\right) = 0 \Rightarrow 3\times\frac{16}{9} - \frac{8}{3}a + b = 0$

$\Rightarrow -8a + 3b = -16$ ...... (2)

$a = 5, b = 8$

SECTION-B

1. Let f(x) be a polynomial of degree 6 in x, in which the coefficient of x^6 is unity and it has extrema at x = –1 and x = 1. If $\lim_{x\to 0}\frac{f(x)}{x^3} = 1$, then $5\cdot f(2)$ is equal to _______.

Ans. 144

Sol. $f(x) = x^6 + ax^5 + bx^4 + x^3$

$\therefore \quad f'(x) = 6x^5 + 5ax^4 + 4bx^3 + 3x^2$

Roots 1 & – 1

$\therefore \quad 6 + 5a + 4b + 3 = 0$ & $-6 + 5a - 4b + 3 = 0$ solving

$a = -\frac{3}{5} \qquad b = -\frac{3}{2}$

$\therefore \quad f(x) = x^6 - \frac{3}{5}x^5 - \frac{3}{2}x^4 + x^3$

$\therefore \quad 5 \,.\, f(2) = 5\left[64 - \frac{96}{5} - 24 + 8\right] = 144$

2. The number of points, at which the function $f(x) = |2x + 1| - 3|x + 2| + |x^2 + x - 2|$, $x \in R$ is not differentiable, is _______.

Ans. 2

Sol. $|2x + 1| - 3|x + 2| + |x + 2|\,|x - 1|$

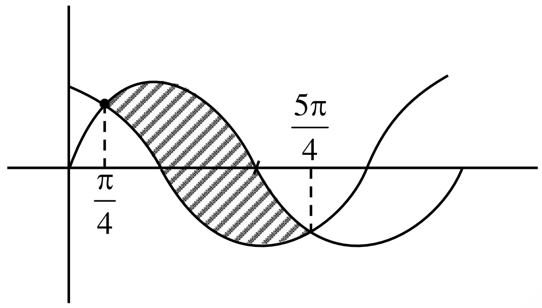
$|2x + 1| - (|x + 2|)(3 - |x - 1|)$

non-differentiable at $x = -\frac{1}{2}, 1$

3. The graphs of sine and cosine functions, intersect each other at a number of points and between two consecutive points of intersection, the two graphs enclose the same area A. Then A^4 is equal to ______.

Ans. 64

Sol.



$$A = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\,dx = \left[-\cos x - \sin x\right]_{\pi/4}^{5\pi/4}$$

$$= -\left[\left(\cos\frac{5\pi}{4} + \sin\frac{\pi}{4}\right) - \left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right)\right]$$

$$= -\left[\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\right] = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

$$\Rightarrow A^4 = (2\sqrt{2})^4 = 64$$

4. Let $A_1, A_2, A_3, \ldots\ldots$ be squares such that for each $n \geq 1$, the length of the side of A_n equals the length of diagonal of A_{n+1}. If the length of A_1 is 12 cm, then the smallest value of n for which area of A_n is less than one, is ______.

Ans. 9

Sol. length of side $A_1 = 12$

length of diagonal $A_2 = 12$

length of side of $A_2 = \frac{12}{\sqrt{2}}$

length side of $A_3 = \frac{12}{(\sqrt{2})^2}$

length of side of $A_n = \frac{12}{(\sqrt{2})^{n-1}}$

Area of $A_n = \left(\frac{12}{(\sqrt{2})^{n-1}}\right)^2 < 1 \Rightarrow \frac{144}{2^{n-1}} < 1 \Rightarrow 144 < 2^{n-1}$

least value of n is 9

5. Let $A = \begin{bmatrix} x & y & z \\ y & z & x \\ z & x & y \end{bmatrix}$, where x, y and z are real numbers such that x + y + z > 0 and xyz = 2.

If $A^2 = I_3$, then the value of $x^3 + y^3 + z^3$ is_____.

Ans. 7

Sol. $A^2 = \begin{bmatrix} x & y & z \\ y & z & x \\ z & x & y \end{bmatrix}\begin{bmatrix} x & y & z \\ y & z & x \\ z & x & y \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow x^2 + y^2 + z^2 = 1$

$\Rightarrow x + y + z = 1$

$\Rightarrow xy + yz + zx = 0$

$|A|^2 = |I| \Rightarrow |A| = \pm 1 \Rightarrow 3xyz - (x^3 + y^3 + z^3) = \pm 1$

$x^3 + y^3 + z^3 = 3.2 \pm 1 = 7, 5$

$\Rightarrow x^3 + y^3 + z^3 = 7$

6. If $A = \begin{bmatrix} 0 & -\tan\left(\frac{\theta}{2}\right) \\ \tan\left(\frac{\theta}{2}\right) & 0 \end{bmatrix}$ and

$(I_2 + A)(I_2 - A)^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, then 13 $(a^2 + b^2)$ is equal to ________ .

Ans. 13

Sol. $A = \begin{bmatrix} 0 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 0 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$I + A = \begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix}, \quad I - A = \begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix}, \quad (I - A)^{-1} = \frac{1}{\sec^2\left(\frac{\theta}{2}\right)}\begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix}$

$\therefore \quad (I + A)(I - A)' = \frac{1}{\sec^2\left(\frac{\theta}{2}\right)}\begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix}\begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix}$

$= \frac{1}{\sec^2\left(\frac{\theta}{2}\right)}\begin{bmatrix} 1 - \tan^2\frac{\theta}{2} & -2\tan\frac{\theta}{2} \\ 2\tan\frac{\theta}{2} & 1 - \tan^2\frac{\theta}{2} \end{bmatrix}$

$$\therefore \quad a = \frac{1-\tan^2\frac{\theta}{2}}{\sec^2\left(\frac{\theta}{2}\right)},\ b = \frac{2\tan\frac{\theta}{2}}{\sec^2\left(\frac{\theta}{2}\right)}$$

$$\therefore \quad 13(a^2+b^2) = 13\frac{1}{\sec^4\left(\frac{\theta}{2}\right)}\left(\left(1-\tan^2\frac{\theta}{2}\right)^2+4\tan^2\frac{\theta}{2}\right) = 13\frac{1}{\sec^4\left(\frac{\theta}{2}\right)}\left(1+\tan^2\frac{\theta}{2}\right)^2 = 13$$

7. The total number of numbers, lying between 100 and 1000 that can be formed with the digits 1, 2, 3, 4, 5, if the repetition of digits is not allowed and numbers are divisible by either 3 or 5, is ______.

Ans. 32

Sol. | | | | divisible by $\rightarrow$ 3

$12 \rightarrow 3, 4, 5 \rightarrow 3! = 6$
$15 \rightarrow 2, 3, 4 \rightarrow 3! = 6$
$24 \rightarrow 1, 3, 5 \rightarrow 3! = 6$
$42 \rightarrow 1, 2, 3 \rightarrow 3! = 6$

24

divisible by 5

| | | 5 | = 12

4×3

Required No. = 24 + 12 – 4 = 32

8. Let $\vec{a} = \hat{i}+2\hat{j}-\hat{k},\ \vec{b} = \hat{i}-\hat{j}$ and $\vec{c} = \hat{i}-\hat{j}-\hat{k}$ be three given vectors. If $\vec{r}$ is a vector such that $\vec{r}\times\vec{a} = \vec{c}\times\vec{a}$ and $\vec{r}\cdot\vec{b} = 0$, then $\vec{r}\cdot\vec{a}$ is equal to ________.

Ans. 12

Sol. $(\vec{r}-\vec{c})\times\vec{a} = \vec{0}$

$\vec{r}-\vec{c} = \lambda\vec{a}$

$\vec{r} = \vec{c}+\lambda\vec{a}$

$\vec{r}\cdot\vec{b} = \vec{c}\cdot\vec{b}+\lambda\vec{a}\cdot\vec{b} = 0$

$= 2-\lambda = 0 \Rightarrow \lambda = 2$

$\vec{r}\cdot\vec{a} = \vec{c}\cdot\vec{a}+\lambda\vec{a}\cdot\vec{a} = 0+2(6)$

9. If the system of equations
$kx + y + 2z = 1$
$3x - y - 2z = 2$
$-2x - 2y - 4z = 3$
has infinitely many solutions, then k is equal to ________.

Ans. 21

Sol. $D = \begin{vmatrix} k & 1 & 2 \\ 3 & -1 & -2 \\ -2 & -2 & -4 \end{vmatrix} = 0$ for $k \in R$

$D_1 = 0$ for $k \in R$

$D_2 = \begin{vmatrix} k & 1 & 2 \\ 3 & 2 & -2 \\ -2 & 3 & -4 \end{vmatrix} = 0$

$\Rightarrow k(-8 + 6) - 1(-12 - 4) + 2(9 + 4) = 0$

$\Rightarrow -2k + 16 + 26 = 0$

$k = 21$

$D_3 = \begin{vmatrix} k & 1 & 1 \\ 3 & -1 & 2 \\ -2 & -2 & 3 \end{vmatrix} = 0$

$\Rightarrow k(-3 + 4) - 1(9 + 4) + 1(-6 - 2) = 0$

$\Rightarrow k = 21$

10. The locus of the point of intersection of the lines $(\sqrt{3})kx + ky - 4\sqrt{3} = 0$ and $\sqrt{3}x - y - 4(\sqrt{3})k = 0$ is a conic, whose eccentricity is ___________.

Ans. 2

Sol. $\sqrt{3}\,x + y - \dfrac{4\sqrt{3}}{k} = 0$

$\sqrt{3}\,x - y - 4\sqrt{3}\,k = 0$

point of intersection

$x = 2\left(k + \dfrac{1}{k}\right) \Rightarrow \dfrac{x}{2} = k + \dfrac{1}{k}$ .... (1)

$y = 2\sqrt{3}\left(\dfrac{1}{k} - k\right) \Rightarrow \dfrac{y}{2\sqrt{3}} = \dfrac{1}{k} - k$ .... (2)

$(1)^2 - (2)^2$

$\dfrac{x^2}{4} - \dfrac{y^2}{12} = 4 \Rightarrow \dfrac{x^2}{16} - \dfrac{y^2}{48} = 1$

eccentricity $= \sqrt{1 + \dfrac{48}{16}} = 2$